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Order of Current Variance and Diffusivity in the Rate One Totally Asymmetric Zero Range Process

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Abstract We prove that the variance of the current across a characteristic is of order $t^{2/3}$ in a stationary constant rate totally asymmetric zero range process, and that the diffusivity has order $t^{1/3}$. This is a step towards proving universality of this scaling behavior in the class of one-dimensional interacting systems with one conserved quantity and concave hydrodynamic flux. The proof proceeds via couplings to show the corresponding moment bounds for a second class particle. We build on the methods developed in Balázs and Seppäläinen (Order of current variance and diffusivity in the asymmetric simple exclusion process, 2006) for simple exclusion. However, some modifications were needed to handle the larger state space. Our results translate into $t^{2/3}$-order of variance of the tagged particle on the characteristics of totally asymmetric simple exclusion.

Keywords Constant rate totally asymmetric zero range process · Diffusivity · Current fluctuations · Second class particle

1 Introduction

The constant rate totally asymmetric zero range process (TAZRP) is a Markov process that describes the motion of particles in the one dimensional integer lattice $\mathbb{Z}$. Namely, if any particles are present at a site, then one of them jumps one site to the right with rate 1 independently of particles at other sites. Later we will give a construction of TAZRP in terms of

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Poisson clocks. This process is among the interacting particle systems introduced in [14]. Our paper examines the net particle current seen by an observer moving at the characteristic speed of the process. The characteristic speed is the speed at which perturbations travel in the system and can be determined e.g. via the hydrodynamic limit. The process is assumed to be in one of its extremal stationary distributions, i.e. with independent geometrically distributed occupation variables at each site with parameter $\frac{1}{1+\varrho}$, implying density $\varrho > 0$ of particles. We prove that the net particle current across the characteristic, that counts the number of particles that pass the observer from left to right minus the number that pass from right to left during time interval $(0, t]$, has variance of order $t^{2/3}$.

The seminal papers of Baik, Deift and Johansson [1] and Johansson [9] gave the first rigorous proofs of such fluctuations in the last-passage version of Hammersley's process and in the totally asymmetric simple exclusion process (TASEP) started from jam-type initial configuration. The correct order was verified to be $t^{1/3}$, and the limiting fluctuations were found to obey Tracy-Widom distributions from random matrix theory. Later a last-passage representation was also found for a stationary TASEP [11], and then the Tracy-Widom limit proved for the current across the characteristic in that setting [8].

Cator and Groeneboom [6] used a probabilistic approach to prove the correct order of these fluctuations for Hammersley's process, Balázs, Cator and Seppäläinen [5] obtained the same result for the last passage representation of the stationary TASEP. Then Balázs and Seppäläinen in [3] proved that the current variance is of order $t^{2/3}$ in the (non-totally) asymmetric simple exclusion process (ASEP).

There are generalizations of the result for the ASEP for those processes in which particles can jump more than one site, most recently by Quastel and Valkó [12, 13].

Physical reasoning, e.g. [15], suggested universality in the sense that similar scaling of the current across the characteristic should occur in other systems as well with one conserved quantity.

To our knowledge, the present article is the first one that generalizes the result of [3] towards another system in which more than one particle per site is allowed. Due to the connection between the TASEP tagged particle and the TAZRP particle current, our results also apply to tagged particle fluctuations of the TASEP. As in [3], our arguments are entirely probabilistic and utilize couplings of several processes and bounds on second class particles. Informally speaking, *second class particles* are perturbations in the system that do not disturb the motion of the regular particles but are influenced by the ambient system. Precise definition will be given after the construction of the coupled processes in Sect. 2. As the state space is larger than in the ASEP, some delicate modifications are required when we couple several processes together.

Fluctuation results for asymmetric exclusion processes have also been stated in terms of a quantity called the *diffusivity* $D(t)$. The link between current variance and diffusivity can be described also using the concept of the second class particle: the variance of the current is the expected absolute deviation of the second class particle, while $tD(t)$ is the variance of the second class particle. For the TAZRP we also obtain the correct order $t^{1/3}$ for the diffusivity.

When the observer's speed V is different from the characteristic speed, the fluctuations are normal, i.e. the current variance is of order t. This was proved by Ferrari and Fontes in [7] for the ASEP and by Balázs for some other processes in [2]. As a consequence we also reproduce these results concerning the normal fluctuations for TAZRP.

This paper is the continuation of [3] and might give some ideas that could lead to the treatment of some other models. The method seems to be robust enough to be able to generalize it to other, even more complicated processes. Proceeding in this direction towards universality is subject to future work.

2 The Constant-Rate Zero Range Process and the Results

2.1 Construction of the Process and the Second Class Particles

The constant-rate totally asymmetric zero range process (TAZRP) is a Markov process on the state space $\Omega = \{0, 1, 2, \ldots\}^{\mathbb{Z}} = \mathbb{Z}^{+\mathbb{Z}}$. Given a state $\underline{\omega} = \{\omega_i\}_{i\in\mathbb{Z}} \in \Omega$, the following jumps can happen independently at different sites:

$$(\omega_i, \omega_{i+1}) \longrightarrow (\omega_i - 1, \omega_{i+1} + 1) \quad \text{with rate } \mathbb{I}_{(\omega_i>0)}. \tag{1}$$

We interpret the process as representing unlabeled particles that execute nearest-neighbor random walks on $\mathbb{Z}$, in such a way that at a site where at least 1 particle is present only the topmost one is allowed to jump to the right with rate 1. The value $\omega_i(t) = 0$ means that site i is vacant at time t. The state of the entire process at time t is then $\underline{\omega}(t)$.

A rigorous construction of this process is done by giving each site i a rate 1 Poisson process $N_{i\to i+1}$ on the time line $[0,\infty)$. The processes $\{N_{i\to i+1}, i \in \mathbb{Z}\}$ are mutually independent, and also independent of the initial configuration $\underline{\omega}(0)$. The rule of evolution is that when $N_{i\to i+1}$ jumps, the topmost particle is moved from i to $i+1$ if i is occupied by at least one particle. Thus the rates (1) are realized.

Let μ_ϱ denote the geometric measure with mean ϱ, i.e.

$$\mu_\varrho\{k\} = \left(\frac{\varrho}{\varrho+1}\right)^k \frac{1}{\varrho+1}$$

on the set $\mathbb{Z}^+$, and let $\underline{\mu}_\varrho = \mu_\varrho^{\otimes\mathbb{Z}}$ be the i.i.d. geometric product measure with marginals μ_ϱ on Ω. It is known that the measures $\{\underline{\mu}_\varrho : 0 \le \varrho\}$ are the extreme points of the convex set of invariant distributions for the process that are also invariant under spatial translations.

It is convenient to embed the zero range process in a height process that represents a wall of adjacent columns of bricks. On top of each interval $[i, i+1]$ sits a column of bricks with height $h_i \in \mathbb{Z}$. The entire height configuration is $\underline{h} = \{h_i\}_{i\in\mathbb{Z}}$, restricted to satisfy

$$0 \le h_{i-1} - h_i \qquad \text{for each } i \tag{2}$$

so that the wall slopes downward to the right. Let the Poisson processes govern the evolution of the heights: when $N_{i\to i+1}$ jumps add a brick on top of the column on $[i, i+1]$. But suppress every step that leads to a violation of (2).

Given an initial particle configuration $\underline{\omega}(0)$, define an initial height configuration by

$$h_i(0) = \begin{cases} \sum_{j=i+1}^{0} \omega_i(0) & \text{for } i < 0, \\ 0 & \text{for } i = 0, \\ -\sum_{j=1}^{i} \omega_i(0) & \text{for } i > 0. \end{cases}$$

Let the heights evolve, and define

$$\omega_i(t) = h_{i-1}(t) - h_i(t).$$

Then this process $\underline{\omega}(t)$ is exactly the TAZRP constructed earlier, and the height increment $h_i(t) - h_i(0)$ is the net particle current across the bond $(i, i+1)$.

Below we run several processes started from different initial configurations together governed by the *basic coupling* in which we use common Poisson clocks for them. The first observation is that this coupling preserves monotonicity among both particle and height configurations. Ordering is defined sitewise: for particle configurations $\underline{\eta} \le \underline{\omega}$ means that $\eta_i \le \omega_i$ for each $i \in \mathbb{Z}$, and similarly for height configurations $\underline{g} \le \underline{h}$ if $g_i \le h_i$ for each $i \in \mathbb{Z}$. The basic coupling has the following property, called attractivity:

$$\underline{\eta}(0) \le \underline{\omega}(0) \Longrightarrow \underline{\eta}(t) \le \underline{\omega}(t) \quad \text{and} \quad \underline{g}(0) \le \underline{h}(0) \Longrightarrow \underline{g}(t) \le \underline{h}(t)$$

for all $t > 0$.

We use the following terminology: if we have two coupled zero range processes $\underline{\eta}(t) \le \underline{\omega}(t)$, then there are $\omega_i - \eta_i$ pieces of $\omega - \eta$ *second class particles* at each site i. The joint process $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$ can be constructed from a two-class process: (i) The first class particles $\underline{\eta}$ obey the TAZRP dynamics as described earlier. (ii) The second class particles also obey the Poisson clocks when they can, but they are only allowed to jump if there are no more first class particles at site i, i.e. their jump rate at site i is $\mathbb{I}_{(\eta_i=0)} \cdot \mathbb{I}_{(\omega_i \ge 1)}$ and only the topmost second class particle is allowed to jump. We emphasize here a different behavior of the zero range second class particles and the ones in the simple exclusion process, i.e. a jump of a first class particle to site i in TAZRP does not affect the number of second class particles at site i. In the ASEP case, where at most one particle is allowed at one site, they interchange sites, implying a left jump of the second class particle even in the totally asymmetric case. Hence the TAZRP model second class particles are allowed to jump only to the right.

Let $\underline{\delta}_i \in \Omega$ denote a configuration that has only a single particle at site i. For $\underline{\eta} \in \Omega$ we can legitimately define $\underline{\eta}^+ = \underline{\eta} + \underline{\delta}_0$. In this situation we say that there is a single second class particle between $\underline{\eta}^+$ and $\underline{\eta}$ at site 0. Since the basic coupling conserves it, there is always a site $Q(t)$ such that

$$\underline{\eta}^+(t) = \underline{\eta}(t) + \underline{\delta}_{Q(t)}.$$

$Q(t)$ is the position of the second class particle at time t, which performs a nearest neighbor walk, influenced by the ambient process $\underline{\eta}(\cdot)$.

It is convenient to also have the notion of a second class *anti*particle at position $Q_a(t)$ in a process $\underline{\omega}(t)$. This means that $Q_a(t)$ is the location of the single discrepancy between two processes $\underline{\omega}(t)$ and $\underline{\omega}^-(t)$ that are started so that $\underline{\omega}^-(0) = \underline{\omega}(0) - \underline{\delta}_i \ge 0$ where $i = Q_a(0)$. In the proofs we will couple more than two processes and this flexibility will be convenient.

2.2 Current Fluctuations and Diffusivity

Let $[x]$ denote the first integer from x towards the origin, in other words $[x] = \lfloor x \rfloor$ (floor) when $x \ge 0$ and $[x] = \lceil x \rceil$ (ceiling) when $x < 0$. For a speed value $V \in \mathbb{R}$ define

$$J^{(V)}(t) = h_{[Vt]}(t), \tag{3}$$

the height of the column over interval $[\,[Vt], [Vt]+1]$ at time t. The normalization $h_0(0) = 0$ implies that $J^{(V)}(t)$ is the total net particle current seen by an observer moving at speed V during time interval $[0, t]$. One can compute the density ϱ stationary expectation

$$\mathbf{E}^{\varrho} J^{(V)}(t) = \frac{\varrho}{\varrho + 1} t - \varrho[Vt] \tag{4}$$

by writing a martingale for $h_0(t)$ and then adding in $h_{[Vt]}(t) - h_0(t)$ which counts particles between sites 0 and $[Vt]$.

Our results are based on an interplay between currents and second class particles. One key fact is the coming connection. To simplify notation we introduce a new measure

$$\hat{\mu}_\varrho(k) = \frac{1}{\mathbf{Var}^\varrho(\omega_0)} \sum_{y=k+1}^{\infty} (y-\varrho)\mu_\varrho(y).$$

It can be checked easily that $\hat{\mu}$ is again a probability measure for any probability measure μ. It arises from [4, Theorem 2.2]. An easy computation gives that in the case of TAZRP $\hat{\mu}_\varrho(k) = (k+1)(\frac{\varrho}{1+\varrho})^k(\frac{1}{1+\varrho})^2$, which is the distribution of the sum of two independent geometrically distributed random variables with mean ϱ. We will use the configuration of a coupled pair of processes started in a product measure where all sites have independent initial marginals μ_ϱ except the origin where the initial distributions are $\hat{\mu}_\varrho$ and $\hat{\mu}_\varrho + 1$ respectively, independently of other sites. Denote this coupling measure of the two states $\underline{\omega}^-(0)$ and $\underline{\omega}(0)$ by $\underline{\hat{\mu}}_\varrho$.

From now on $\mathbf{E}^\varrho$ and $\mathbf{Var}^\varrho$ denotes expectation and variance in the stationary process with density ϱ (process started from $\mu_\varrho^{\otimes\mathbb{Z}}$) while $\mathbf{E}$ and $\mathbf{Var}$ denotes expectation and variance in a coupled pair of processes started from $\underline{\hat{\mu}}_\varrho$.

Proposition 1 *Let $\underline{\omega}(\cdot)$ and $\underline{\omega}^-(\cdot)$ be a coupled pair of TAZRP started from the distribution $\underline{\hat{\mu}}_\varrho$. Let $Q_a(\cdot)$ be a second class antiparticle between the processes starting at $Q_a(0) = 0$. Let the model evolve according to the basic coupling. Then the variance of the current for any $V \in \mathbb{R}$ in the stationary process can be rewritten as*:

$$\begin{aligned}\mathbf{Var}^\varrho(J^{(V)}(t)) &= \mathbf{Var}^\varrho(\omega_0)\mathbf{E}(|[Vt] - Q_a(t)|) \\ &= \varrho(1+\varrho)\mathbf{E}(|[Vt] - Q_a(t)|). \end{aligned} \tag{5}$$

Also,

$$\mathbf{E}\big(Q_a(t)\big) = V^\varrho t = \frac{1}{(1+\varrho)^2}t \tag{6}$$

holds.

In article [4] identities (5) and (6) are proved for a class of attractive models that include the asymmetric simple exclusion, zero-range and bricklayer processes.

The interesting current fluctuations occur at the *characteristic speed* $V^\varrho = \frac{1}{(1+\varrho)^2}$. From (6) we see that this is the average speed of the second class particle. The characteristic speed also appears in the PDE theory of the conservation law

$$\varrho_t + f(\varrho)_x = 0 \tag{7}$$

that is obtained via the Eulerian hydrodynamic limit of TAZRP. Here the hydrodynamic flux takes the form

$$f(\varrho) = \mathbf{E}^\varrho\big(\mathbb{I}_{(\omega>0)}\big) = \frac{\varrho}{1+\varrho}. \tag{8}$$

At constant density ϱ the characteristic speed is $f'(\varrho) = V^\varrho$ at which perturbations of entropy solutions of (7) travel. For more information we refer to [10] for hydrodynamic limits of interacting particle systems.

Let us return to the stationary TAZRP with Geometric ($\frac{1}{1+\varrho}$) occupation variables at each fixed time. A basic object for understanding space-time correlations is the *two point function* $S(i,t) = \mathbf{E}^\varrho(\omega_i(t)\omega_0(0)) - \varrho^2$. Due to [4] it can be written as the transition probability of the second class particle:

$$S(i,t) = \mathbf{E}^\varrho\left(\mathbb{I}\{Q(t)=n\}\cdot\sum_{z=\omega_0+1}^{\infty}(z-\varrho)\frac{\mu_\varrho(z)}{\mu_\varrho(\omega_0)}\right) = \mathbf{Var}^\varrho(\omega_0)\cdot\mathbf{P}\big\{Q(t)=n\big\}. \tag{9}$$

The sum under the expectation is regarded as a Radon-Nikodym derivative that takes our second class particle in the initial setting $\underline{\hat{\mu}}_\varrho$, where the second class particle resides initially at the origin. Note that use of the term "transition probability" is not meant to suggest that $Q(t)$ is a Markov process.

From (9) and (6) follows

$$\sum_{i\in\mathbb{Z}} iS(i,t) = \mathbf{Var}^\varrho(\omega_0)V^\varrho t = \varrho(1+\varrho)V^\varrho t.$$

The *diffusivity* is by definition a normalized second moment of the two-point function:

$$D(t) = \frac{1}{t\varrho(1+\varrho)}\sum_{i\in\mathbb{Z}}(i-V^\varrho t)^2 S(i,t).$$

Consequently the diffusivity can also be expressed in terms of the variance of the second class particle:

$$D(t) = t^{-1}\mathbf{Var}(Q(t)).$$

Below we show that, similarly to the ASEP, the TAZRP second class particle is also *superdiffusive* with variance of order $t^{4/3}$.

We can now state the main theorem, a moment bound for the second class particle in a pair started from $\underline{\hat{\mu}}_\varrho$.

Theorem 2 *For the constant rate TAZRP $\underline{\omega}(\cdot)$ started in $\underline{\hat{\mu}}_\varrho$ distribution for any $\varrho \ge 0$ there exist constants $0 < t_0, C < \infty$ such that for real $1 \le m < 3$ and $t \ge t_0$*

$$C^{-1} \le \mathbf{E}\left\{\left|\frac{Q_a(t) - V^\varrho t}{t^{2/3}}\right|^m\right\} \le C. \tag{10}$$

Let us simplify notation to $J^\varrho(t) = J^{(V^\varrho)}(t)$ for the current across the characteristic. Via (5), taking $m = 1$ gives the variance of this current.

Corollary 3 *With assumptions as in Theorem 2, for $t \ge t_0$ the variance of the current across the characteristic satisfies*

$$C^{-1}t^{2/3} \le \mathbf{Var}^\varrho(J^\varrho(t)) \le Ct^{2/3}.$$

Taking $m = 2$ identifies the order of the diffusivity.

Corollary 4 *With assumptions as in Theorem 2, for $t \ge t_0$*

$$C^{-1}t^{1/3} \le D(t) \le Ct^{1/3}.$$

The upper bound of (10) is not valid for all $t > 0$. For small t the variance of $Q(t)$ is of order t because the second class particle is likely to have experienced at most one jump.

The above results are completely analogous to those in [3].

Next, another important consequence of the theorem is the Weak Law of Large Numbers for $Q(t)$. Markov's inequality implies

$$\mathbf{P}\left(\left|\frac{Q(t)}{t} - V^{\varrho}\right| > \varepsilon\right) \le \frac{\mathbf{E}|Q(t) - [V^{\varrho}t]| + 1}{\varepsilon t} \le \frac{Ct^{2/3}}{\varepsilon t} \to 0.$$

We mention here that the moment bounds of (10) together with the fact that $Q(t)$ is dominated by a Poisson process are strong enough to derive the Strong Law of Large Numbers as well.

Proposition 1 and Theorem 2 imply that the current variance is of order t for all $V \ne V^{\varrho}$:

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathbf{Var}^{\varrho} J^{(V)}(t)}{t} &= \lim_{t\to\infty} \frac{1}{t}\mathbf{Var}^{\varrho}(\omega_0)\mathbf{E}(|Q(t) - [Vt]|)\\ &= \lim_{t\to\infty} \frac{1}{t}\mathbf{Var}^{\varrho}(\omega_0)\big(\mathbf{E}|Q(t) - [V^{\varrho}t] + [V^{\varrho}t] - [Vt]|\big)\\ &= \mathbf{Var}^{\varrho}(\omega_0)|V^{\varrho} - V|.\end{aligned}$$

The Central Limit Theorem for $J^{(V)}(t)$ also holds in the following form:

$$\mathbf{P}\left(\frac{J^{(V)}(t) - \mathbf{E}^{\varrho} J^{(V)}(t)}{\sqrt{t\mathbf{Var}^{\varrho}(\omega_0)|V^{\varrho} - V|}} < x\right) \to \Phi(x) \quad \text{for all } x \in \mathbb{R},\ V \ne V^{\varrho},$$

where $\Phi(x)$ denotes the standard normal distribution.

A more important consequence of Theorem 2 concerns the fluctuations of the tagged particle of the TASEP near the characteristics. To be able to explain the result we introduce some new notation: first, we consider a stationary TASEP with independent Bernoulli(α) occupation variables at each site which we condition on the event that the origin is occupied. Next we label the particles in an increasing order such a way that the one at the origin becomes label 0. We denote the site of the i-th particle at time t by $R_i^E(t)$. Then we let the system evolve according to the TASEP evolution with particles jumping to the left and not to the right. Rates of these jumps are $\mathbb{I}_{\{\omega_i^E=1\}} \cdot \mathbb{I}_{\{\omega_{i-1}^E=0\}}$, i.e. the particles are allowed to jump one to the left if the neighboring site on the left is empty. The particles in the TASEP cannot pass each other, hence the ordering is preserved over time. As we conditioned on the origin being occupied, $R_0^E(0) = 0$ and we choose the usual height representation of the process with $h_0^E(0) = 0$ and the height sloping to the left. Since the particles jump to the left, this height function is decreasing in time.

Second, we look at the process of the differences, namely the empty sites between two neighboring particles in the exclusion process. Define $\omega_i(t) := R_i^E(t) - R_{i-1}^E(t) - 1$. Then the process $\underline{\omega}(t)$ is a stationary TAZRP with parameter $\varrho = 1/\alpha - 1$, where particles jump to the right. The height representation $\underline{h}(t)$ of the zero range process can be translated to the TASEP, namely

$$R_k^E(t) = -h_k(t) + k. \tag{11}$$

For $k = [V^{\varrho}t]$, we get $R_{[V^{\varrho}t]}^E(t) = -J^{\varrho}(t) + [V^{\varrho}t]$. Theorem 2 leads to

$$C^{-1} \le \liminf_{t\to\infty} \frac{\mathbf{Var}(R_{[V^{\varrho}t]}^E(t))}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(R_{[V^{\varrho}t]}^E(t))}{t^{2/3}} \le C$$

To give an interpretation of this result, we recall that the characteristic speed in the TAZRP equals $V^{\varrho} = \frac{1}{(1+\varrho)^2} = \alpha^2$, so (11) at site $[V^{\varrho}t]$ turns into $R_{[\alpha^2 t]}(t) = -J^{\varrho}(t) + [\alpha^2 t]$. Since the mean distance between particles in the TASEP is $1/\alpha$, the $[\alpha^2 t]^{\text{th}}$ particle started at time 0 somewhere near αt. Burke's Theorem implies that each particle marginally moves to the left following a Poisson process with parameter $1-\alpha$, hence this particle should be at time t somewhere near $(2\alpha - 1)t$, which is exactly the characteristic position of the exclusion process. So, to put the result $\mathbf{Var}(R^E_{[\alpha^2 t]}) \sim t^{2/3}$ in words we can say that the position of the tagged particle that is expected to be on the characteristics has fluctuation of order $t^{1/3}$.

The rest of the paper is devoted to the proof of Theorem 2, first the upper bound and then the lower bound.

3 Upper Bound

Proof of the upper bound utilizes couplings of several processes. Before deriving the upper bound we introduce some preliminaries on the couplings.

3.1 Couplings

We start by describing an initial distribution of three coupled processes and with labels attached to second class particles between two of the three processes.

Definition 5 For two probability measures $\mu_1 \leq \mu_2$ denotes stochastic domination.

Equivalent to stochastic domination are:

- For the cumulative distribution functions $F_1(x) \geq F_2(x)$ holds for all $x \in \mathbb{R}$.
- There exists a coupling measure $\mu(x, y)$ with marginals μ_1 and μ_2 such that $\mu\{(x, y) : x \leq y\} = 1$.

Fix densities $0 < \lambda < \varrho$. In the next sections we want to couple models with different densities and distribution so here we prove stochastic domination of the different measures.

Lemma 6 *The measures μ_ϱ and $\hat{\mu}_\varrho$ are monotone in ϱ, i.e.*

1. $\mu_\lambda \leq \mu_\varrho$ *for all* $\lambda \leq \varrho$,
2. $\hat{\mu}_\lambda \leq \hat{\mu}_\varrho$ *for all* $\lambda \leq \varrho$
3. *Furthermore,* $\hat{\mu}_\varrho \geq \mu_\varrho$ *for all* $\varrho > 0$.

Proof The cumulative distribution function of the geometric distribution with mean ϱ is

$$F_\varrho(z) = \sum_{k=0}^{z} \mu_\varrho(k) = \sum_{k=0}^{z} \frac{\varrho^k}{(1+\varrho)^{k+1}} = 1 - \left(\frac{\varrho}{1+\varrho}\right)^{z+1}.$$

Differentiation with respect to ϱ gives that the function is monotone decreasing in ϱ, meaning that $\lambda \leq \varrho$ implies $F_\varrho(z) \leq F_\lambda(z)$. Next, $\hat{\mu}_\varrho$ is the distribution of the sum of two independent, geometrically distributed random variables, implying $\widehat{F}_\varrho(z) \leq \widehat{F}_\lambda(z)$. The third statement comes from the fact that a random variable of distribution $\hat{\mu}_\varrho$ is the sum of two independent random variables of distribution μ_ϱ. □

Definition 7

- ν is the coupling measure with marginals μ_λ and μ_ϱ with $\nu\{(\eta,\omega):\eta\le\omega\}=1$. Let $\underline{\nu}:=\nu^{\otimes\mathbb{Z}}$ be the product measure with marginals ν for each site.
- $\hat{\nu}$ is the coupling measure with marginals $\hat{\mu}_\lambda$ and $\hat{\mu}_\varrho$ with $\hat{\nu}\{(\eta,\omega):\eta\le\omega\}=1$.

We now describe the initial distribution. Let $(\underline{\eta},\underline{\omega}^-)$ be distributed in the product measure of marginals ν for each site $i\neq 0$ and $\hat{\nu}$ at the origin. We also couple a third process denoted by $\underline{\omega}$ to these two processes with $\omega(0)=\omega^-(0)+\delta_0$ i.e. we add an extra second class particle to the process $\underline{\omega}^-$. Saying differently, we have a second class antiparticle on $\underline{\omega}$. Let $\underline{\bar{\nu}}$ denote the resulting joint distribution of $(\underline{\eta},\underline{\omega}^-,\underline{\omega})$. Then $\underline{\bar{\nu}}$-a.s. we have the following initial conditions for the processes $\underline{\eta}$, $\underline{\omega}^-$ and $\underline{\omega}$:

- $\underline{\eta}\le\underline{\omega}^-\le\underline{\omega}$,
- There is a second class particle between $\underline{\omega}$ and $\underline{\omega}^-$ at the origin at time 0, implying that there is at least one second class particle between $\underline{\omega}$ and $\underline{\eta}$ at the origin.
- There are infinitely many $\omega-\eta$ second class particles on both sides of the origin.

Label the $\omega-\eta$ second class particles with integers in an increasing fashion from left to right, giving label 0 to the topmost second class particle initially at the origin. $X_k(0)$ denotes the position of second class particle with label k, so that

$$\cdots\le X_{-1}(0)\le X_0(0)=0<X_1(0)\le\cdots \tag{12}$$

Let these configurations evolve from the initial distribution $\underline{\bar{\nu}}$ with common Poisson clocks. The rule for the labeling is that whenever a second class particle jumps from site i, it is always the highest labeled one at i and it arrives at the lowest place at site $i+1$. This way the ordering (12) is kept for all time, and the basic coupling preserves the ordering $\underline{\eta}(t)\le\underline{\omega}^-(t)\le\underline{\omega}(t)$ a.s. for all times t. The topmost $\omega-\eta$ second class particle that starts at the origin is of special importance to us, so we set $X(t)=X_0(t)$. The effect of the coupling is that the $\omega-\omega^-$ particle (denote its position from now on by $Q_a(t)$) cannot pass $X(t)$. Precisely saying, the ordering $Q_a(t)\le X(t)$ is preserved over time. The argument for this fact is that initially $Q_a(0)\le X(0)$ holds. Since both particles are only allowed to jump to the right Q_a can pass X if and only if they are at the same site. In this case $X(t)$ jumps if $\eta_{X(t)}(t)=0$ but $Q_a(t)$ jumps only if $\omega^-_{Q_a(t)}=0$, and the coupling $\underline{\eta}\le\underline{\omega}^-$ guarantees that the second event is part of the first one, i.e. the jump of $Q_a(t)$ implies the jump of $X(t)$ but not conversely.

Throughout this section probability $\mathbf{P}$ refers to the law of the three processes started in distribution $\underline{\hat{\nu}}$, evolving according to the basic coupling as described above.

3.2 Proof of the Upper Bound

We begin with the proof of the upper bound in Theorem 2. C and its variants $C_1, C_2,\ldots$ denote positive constants that possibly depend on ϱ and whose values can change from line to line. We first prove a lower bound on the density of the $\omega-\eta$ second class particles. For integers $j\in\mathbb{Z}$ and $u>0$ let

$$N_j(t)=\sum_{i=j+1}^{j+2u-1}(\omega_i(t)-\eta_i(t)). \tag{13}$$

Lemma 8 *Let $\varrho > \lambda > 0$ and $d \geq 0$ be an integer. Then there are strictly positive finite constants $\gamma = \gamma(\varrho)$, $C_1 = C_1(\varrho, d)$ and $C_2 = C_2(\varrho)$ such that the following holds: if $0 < \varrho - \lambda < \gamma$, then for all integers $j \in \mathbb{Z}$, $u > 0$ and any time $t \geq 0$,*

$$\mathbf{P}\{N_j(t) < u(\varrho - \lambda) + d\} \leq C_1 \exp\{-C_2 u(\varrho - \lambda)^2\}$$

Proof For the moment, denote by $\underline{y}(\cdot)$ a constant-rate TAZRP such that $y_i(0) = \omega_i(0)$ at all sites i except for $i = 0$, and $\underline{z}(\cdot)$ is a constant-rate TAZRP process such that $z_i(0) = \eta_i(0)$ at all sites i except for $i = 0$. For $i = 0$, we pick the pair $(z_0(0), y_0(0))$ in distribution ν, independently of the configuration on other sites, such that $z(0) \leq y(0) \leq \omega^-(0)$ and $z(0) \leq \eta(0) \leq \omega^-(0)$ holds. This can be done since $\mu_\lambda \leq \mu_\varrho \leq \hat{\mu}_\varrho$ and $\mu_\lambda \leq \hat{\mu}_\lambda \leq \hat{\mu}_\varrho$. Apply the basic coupling to ensure $\underline{z}(t) \leq \underline{\eta}(t) \leq \underline{\omega}(t)$ and $\underline{y}(t) \leq \underline{\omega}^-(t)$ for all $t \geq 0$ (notice that this holds initially). $\underline{y}(t)$ and $\underline{z}(t)$ are marginally time-stationary processes, hence we can omit the notation for their time dependence in our arguments. However, the pair $(\underline{z}(t), \underline{y}(t))$ is *not* in product distribution for $t > 0$. Define

$$Y = \sum_{i=j+1}^{j+2u-1} y_i \quad \text{and} \quad Z = \sum_{i=j+1}^{j+2u-1} z_i,$$

so that $N_j(t) = \sum_{i=j+1}^{j+2u-1}(\omega_i(t) - y_i(t)) + Y - Z + \sum_{i=j+1}^{j+2u-1}(z_i(t) - \eta_i(t))$. In this expression the two sums can be estimated from above by the total number of $\omega - y$ and $z - \eta$ second class particles, respectively. Using the facts that these models only differ initially at the origin and the total number of second class particles is preserved over time by the basic coupling allows us to use the estimations $0 \leq \sum_{i=j+1}^{j+2u-1}(\omega_i(t) - y_i(t))$ and $\sum_{i=j+1}^{j+2u-1}(z_i(t) - \eta_i(t)) \geq z_0(0) - \eta_0(0) \geq -\eta_0(0)$. If we plug in the lower bounds the probability of the event increases:

$$\mathbf{P}\big\{N_j(t) < u(\varrho - \lambda) + d\big\} \leq \mathbf{P}\big\{Y - Z - \eta_0(0) < u(\varrho - \lambda) + d\big\}$$

Now we use exponential Markov's inequality to get for any $\alpha > 0$

$$\begin{aligned}
&\mathbf{P}\big\{N_j(t) < u(\varrho - \lambda) + d\big\}\\
&\quad\leq \mathbf{P}\left\{\mathrm{e}^{-\alpha(Y - Z - \eta_0(0))} > \mathrm{e}^{-\alpha u(\varrho-\lambda) - d\alpha}\right\}\\
&\quad\leq \mathrm{e}^{\alpha u(\varrho-\lambda)+d\alpha} \cdot \mathbf{E}\big(\mathrm{e}^{-\alpha Y}\mathrm{e}^{\alpha Z}\mathrm{e}^{\alpha\eta_0(0)}\big)\\
&\quad\leq \mathrm{e}^{\alpha u(\varrho-\lambda)+d\alpha} \cdot \big[\mathbf{E}^\varrho\big(\mathrm{e}^{-4\alpha Y}\big)\big]^{1/4} \cdot \big[\mathbf{E}^\lambda\big(\mathrm{e}^{4\alpha Z}\big)\big]^{1/4} \cdot \big[\mathbf{E}\big(\mathrm{e}^{2\alpha\eta_0(0)}\big)\big]^{1/2}\\
&\quad= \mathrm{e}^{\alpha u(\varrho-\lambda)+d\alpha} \cdot \big[\mathbf{E}^\varrho\big(\mathrm{e}^{-4\alpha y_0}\big)\big]^{1/2u-1/4} \cdot \big[\mathbf{E}^\lambda\big(\mathrm{e}^{4\alpha z_0}\big)\big]^{1/2u-1/4} \cdot \big[\mathbf{E}\big(\mathrm{e}^{2\alpha\eta_0(0)}\big)\big]^{1/2}\\
&\quad= \exp\left\{\alpha u(\varrho - \lambda) + d\alpha - \left(\frac{1}{2}u - \frac{1}{4}\right)\log\big[1 + \varrho(1 - \mathrm{e}^{-4\alpha})\big]\right.\\
&\qquad\left. - \left(\frac{1}{2}u - \frac{1}{4}\right)\log\big[1 + \lambda(1 - \mathrm{e}^{4\alpha})\big] - \log\big[1 + \lambda(1 - \mathrm{e}^{2\alpha})\big]\right\}\\
&\quad\leq \exp\{-\alpha u(\varrho - \lambda) + 4u(\varrho + \lambda + \varrho^2 + \lambda^2)\alpha^2\\
&\qquad + uC_3\alpha^3 + C_4\alpha + C_5\alpha^2 + \mathcal{O}(\alpha^3)\}.
\end{aligned}$$

Here we used the marginal $\underline{\mu}_\lambda$ and $\underline{\mu}_\varrho$ product distributions of $\underline{z}$ and $\underline{y}$, and the fact that $\eta_0(0) \sim \hat{\mu}_\lambda$ which implies $\mathbf{E}(\mathrm{e}^{2\alpha\eta_0(0)}) = (1+\lambda+\lambda \mathrm{e}^{2\alpha})^{-2}$. The last inequality comes from Taylor expansion w.r.t. α. The $\mathcal{O}(\alpha^3)$ term is uniform over $\lambda < \varrho$ for a fixed ϱ. In order to ensure the existence of the exponential moment we need the condition $\frac{\lambda}{1+\lambda}e^{2\alpha} < 1$ to be satisfied, or equivalently, $\alpha < \frac{1}{2}\log(\frac{1+\lambda}{\lambda})$. Next we pick

$$\alpha = \frac{\varrho-\lambda}{8(\varrho+\lambda+\varrho^2+\lambda^2)},$$

optimizing the ϱ and λ-dependent terms in which u is also present. Comparing this to the required condition above gives a bound for γ, i.e. $\gamma \le 4\log(\frac{1+\varrho}{\varrho})\cdot(\varrho+\varrho^2)$. By this choice of α we get

$$\mathbf{P}\big\{N_j(t) < u(\varrho-\lambda)+d\big\} \le e^{\alpha(d+2\lambda)+\alpha^2 C_3+\mathcal{O}(\alpha^3)} \exp\left\{-\frac{u(\varrho-\lambda)^2}{16(\varrho+\lambda+\varrho^2+\lambda^2)}\right\}$$

Now $\varrho-\lambda \le \gamma$ gives rise to the constant C_1 in the first factor and finishes the proof. □

Now we are ready to turn to the main estimate for the upper bound. The idea is to bound the deviation $\mathbf{P}\{Q_a(t) \ge u + [V^\varrho t]\}$ with an appropriate expression that involves the moment $\mathbf{E}|Q_a(t) - [V^\varrho t]|$. This is completed in (18) below and then the upper bound comes from an elementary integration step.

Along the way we compare currents in two processes that we abbreviate as follows:

$$J^\varrho(t) = J^{(\frac{1}{1+\varrho^2})}(t) \quad \text{for current in the } \underline{\omega}(.) \text{ process, and}$$
$$J^{V^\varrho,\lambda}(t) = J^{(\frac{1}{1+\varrho^2})}(t) \quad \text{for current in the } \underline{\eta}(.) \text{ process.}$$

Notice that both use the same speed $V^\varrho = f'(\varrho) = \frac{1}{1+\varrho^2}$ for the observer. As already defined in the Introduction, this is the characteristic speed of the $\underline{\omega}(.)$ process with density ϱ. From now on we denote by tilde the centered random variable.

Lemma 9 *Suppose $\varrho-\lambda < \gamma$ with γ from Lemma 8. Then for positive integers u and times $t \in [0,\infty)$,*

$$\begin{aligned}\mathbf{P}\{Q_a(t) \ge 2u + [V^\varrho t]\} \\ \le \mathbf{P}\left\{\widetilde{J}^\varrho(t) - \widetilde{J}^{V^\varrho,\lambda}(t) \ge u(\varrho-\lambda) - t\frac{(\varrho-\lambda)^2}{(1+\varrho)^2}\right\} \\ + C_1\exp\{-C_2 u(\varrho-\lambda)^2\}.\end{aligned} \tag{14}$$

Proof Now we recall the construction of the labeled particles and the fact that $Q_a(t) \le X(t)\ \forall t$:

$$\mathbf{P}\{Q_a(t) \ge 2u + [V^\varrho t]\} \le \mathbf{P}\{X(t) \ge 2u + [V^\varrho t]\}.$$

$N_{[V^\varrho t]}(t)$ of (13) counts the number of $\omega - \eta$ second class particles at time t in the interval $\{[V^\varrho t]+1,\dots,[V^\varrho t]+2u-1\}$. Since the second class particles stay ordered and $X(t)$ started at the origin, the event $\{X(t) \ge 2u+[V^\varrho t]\}$ implies that all these second class particles crossed the path $s \mapsto [V^\varrho s]+1/2$ by time t. Each such second class particle crossing increases $J^{\varrho}(t) - J^{V^{\varrho},\lambda}(t)$ by one. Therefore

$$\begin{aligned}\mathbf{P}\{X(t) \geq 2u + [V^{\varrho}t]\} &\leq \mathbf{P}\{J^{\varrho}(t) - J^{V^{\varrho},\lambda}(t) \geq N_{[V^{\varrho}t]}(t)\}\\ &\leq \mathbf{P}\big\{J^{\varrho}(t) - J^{V^{\varrho},\lambda}(t) \geq u(\varrho - \lambda) + 4\varrho + 1\big\}\\ &\quad + \mathbf{P}\big\{N_{[V^{\varrho}t]}(t) < u(\varrho - \lambda) + 4\varrho + 1\big\}.\end{aligned}$$

Combine the previous displays to get

$$\mathbf{P}\{Q_a(t) \geq 2u + [V^{\varrho}t]\} \leq \mathbf{P}\big\{J^{\varrho}(t) - J^{V^{\varrho},\lambda}(t) \geq u(\varrho - \lambda) + 4\varrho + 1\big\} \tag{15}$$

$$+ \mathbf{P}\big\{N_{[V^{\varrho}t]}(t) < u(\varrho - \lambda) + 4\varrho + 1\big\} \tag{16}$$

To line (16) apply Lemma 8 with $d = \lceil 4\varrho + 1\rceil$. We see that line (16) is bounded by the last exponential term in (14). Recall the definition of the flux (8). If $\underline{\omega}$ and $\underline{\eta}$ start from their respective $\underline{\mu}_{\varrho}$ and $\underline{\mu}_{\lambda}$ equilibria then we would have, due to (4) and (6),

$$\begin{aligned}\mathbf{E}^{\varrho}(J^{\varrho}(t)) - \mathbf{E}^{\lambda}(J^{V^{\varrho},\lambda}(t)) &= t\big(f(\varrho) - f(\lambda) - f'(\varrho)(\varrho - \lambda)\big)\\ &= t\frac{(\varrho - \lambda)^2}{(1+\varrho)^2(1+\lambda)}\end{aligned}$$

where we ignored the error coming from the integer part of $V^{\varrho}t$. This error can be estimated from above by the number of particles sitting at one site in each of the processes, i.e. $\mathbf{E}^{\varrho}(J^{\varrho}(t)) - \mathbf{E}^{\lambda}(J^{V^{\varrho},\lambda}(t)) - t\frac{(\varrho-\lambda)^2}{(1+\varrho)^2(1+\lambda)} \leq \mathbf{E}^{\varrho}(\omega_i) + \mathbf{E}^{\lambda}(\eta_i) = \varrho + \lambda$. Our processes are also perturbed initially at the origin by $\hat{\nu}$ being different of ν, which gives an error $\mathbf{E}(J^{\varrho}(t)) - \mathbf{E}^{\varrho}(J^{\varrho}(t)) \leq \mathbf{E}(\omega_0(0)) = 2\varrho + 1$. In the $\underline{\eta}$ process the term $-\mathbf{E}(J^{V_{\varrho},\lambda}(t)) + \mathbf{E}^{\lambda}(J^{V_{\varrho},\lambda}(t))$ is negative so it can be estimated from above by 0. The term $4\varrho + 1$ inside the probability on line (15) makes up for these errors. So, we get $\mathbf{P}\{\widetilde{J}^{\varrho}(t) - \widetilde{J}^{V^{\varrho},\lambda}(t) \geq u(\varrho - \lambda) - t\frac{(\varrho-\lambda)^2}{(1+\varrho)^2(1+\lambda)}\}$. Finally, we omit the factor $1 + \lambda$ in the denominator, implying the decrease of the right hand side of the inequality, and increasing the probability. □

Lemma 10

$$\begin{aligned}\mathbf{Var}(J^{\varrho}(t)) &\leq 12\varrho(\varrho + 1) + 2 + 2\varrho(1 + \varrho)\mathbf{E}(|[V^{\varrho}t] - Q_a(t)|),\\ \mathbf{Var}(J^{V^{\varrho},\lambda}(t)) &\leq 12\varrho(\varrho + 1) + 4t(\varrho - \lambda) + 2\varrho(1 + \varrho)\mathbf{E}(|[V^{\varrho}t] - Q_a(t)|).\end{aligned}$$

Proof The variance **Var** in the statement is taken in the three-process coupling where the geometric distribution $\underline{\mu}_{\varrho}$ is initially perturbed at the origin. Recall that $\mathbf{Var}^{\varrho}$ denotes variance in the stationary process $\underline{y}(t)$ coupled to $\underline{\omega}(t)$, and denote by $J^{\varrho}_y(t)$ the height function in this process at site $[V^{\varrho}t]$ at time t. The following estimation holds:

$$\begin{aligned}\mathbf{Var}(J^{\varrho}(t)) &\leq 2\mathbf{Var}(J^{\varrho}(t) - J^{\varrho}_y(t)) + 2\mathbf{Var}^{\varrho}(J^{\varrho}(t))\\ &\leq 2\mathbf{E}(\omega_0(0)^2) + 2\mathbf{Var}^{\varrho}(\omega_0(0)) \cdot \mathbf{E}(|Q_a(t) - [V^{\varrho}t]|)\\ &\leq 12\varrho(\varrho + 1) + 2 + 2\varrho(\varrho + 1)\mathbf{E}(|Q_a(t) - [V^{\varrho}t]|).\end{aligned} \tag{17}$$

Here we applied Proposition 1 to the second term $\mathbf{Var}^{\varrho}(J^{\varrho}(t))$ in the first line and used the fact that $J^{\varrho}(t)$ and $J^{\varrho}_y(t)$ only differ at the origin by $\omega_0(0) - y_0(0)$, so its variance can be estimated from above by the second moment of $\omega_0(0)$, equal to the second moment of 1 plus the sum of two independent geometrically distributed random variables with mean ϱ. Similar computation for the λ-density process is as follows (with stationary λ-density process $\underline{z}(t)$ and its height function $J_{\underline{z}}^{V^\varrho,\lambda}(t)$):

$$\begin{aligned}
\mathbf{Var}(J^{V^\varrho,\lambda}(t)) &\le 2\mathbf{Var}(J^{V^\varrho,\lambda}(t) - J_{\underline{z}}^{V^\varrho,\lambda}(t)) + 2\mathbf{Var}^\lambda(J^{V^\varrho,\lambda}(t))\\
&\le 2\mathbf{E}(\eta_0(0)^2) + 2\mathbf{Var}^\lambda(\eta_0(0)) \cdot \mathbf{E}(|Q^\lambda(t) - [V^\varrho t]|)\\
&\le 4\lambda(\lambda+1) + 8\lambda^2 + 2\lambda(\lambda+1)\mathbf{E}(|Q^\lambda(t) - [V^\varrho t]|)\\
&\le 12\varrho(\varrho+1) + 2\lambda(\lambda+1)\mathbf{E}(|Q^\lambda(t) - Q_a(t)|)\\
&\quad + 2\varrho(\varrho+1)\mathbf{E}(|Q_a(t) - [V^\varrho t]|)\\
&\le 12\varrho(\varrho+1) + 4(\varrho-\lambda)t + 2\varrho(\varrho+1)\mathbf{E}(|Q_a(t) - [V^\varrho t]|).
\end{aligned}$$

In the first line we cut the variance into two terms, where the first term can be estimated from above by $2\mathbf{E}(\eta_0(0)^2)$, and for the second term we applied Proposition 1 with a second class particle $Q^\lambda(t)$ added to the $\underline{\eta}$ process. Then we used triangle inequality and Proposition 1 again to get $\mathbf{E}(|Q^\lambda(t) - Q_a(t)|) = \mathbf{E}(Q^\lambda(t) - Q_a(t)) = t\frac{1}{(1+\lambda)^2} - t\frac{1}{(1+\varrho)^2}$. The absolute value can be omitted by $Q^\lambda(t) \ge Q_a(t)$, which we show below. Initially $Q^\lambda(0) = Q_a(0) = 0$. The jump rates of $Q^\lambda(t)$ and $Q_a(t)$ are $\mathbb{I}_{\eta_{Q^\lambda(t)}(t)=0}$ and $\mathbb{I}_{\omega^-_{Q_a(t)}(t)=0}$, respectively. Second class particles are only allowed to jump to the right, and a right jump of Q_a without Q^λ is impossible by the basic coupling and the above rates since $\eta_i(t) \le \omega_i^-(t)$.

Some algebra then leads to the estimation $2\lambda(\lambda+1)\mathbf{E}(|Q^\lambda(t) - Q_a(t)|) \le 4(\varrho-\lambda)t$. Collecting terms completes the proof of the lemma. □

We come to the lemma that summarizes all the previous estimations.

Lemma 11 *For any real $u \ge 1$ and time $t > 0$,*

$$\begin{aligned}
\mathbf{P}\{Q_a(t) \ge 2u + [V^\varrho t]\} &\le C_3\frac{t^2}{u^4} \cdot \mathbf{E}\big(|Q_a(t) - [V^\varrho t]|\big)\\
&\quad + C_4\frac{t^2}{u^3} + C_5\frac{t^2}{u^4} + C_1\exp\left\{-C_2\frac{u^3}{t^2}\right\} + \mathrm{e}^{-u}.
\end{aligned} \tag{18}$$

Proof Set $b = \frac{2}{(1+\varrho)^2}(\gamma \wedge \varrho)$ where γ is the constant from Lemma 8. We proceed by considering three cases.

Case 1 $1 \le u < bt$. Suppose first u is an integer as it was in the proof of (14). Throughout density ϱ has been fixed, and now we also fix

$$\lambda = \varrho - \frac{u}{2t}(1+\varrho)^2.$$

The constraint on u guarantees that $\lambda > 0$ and $\varrho - \lambda < \gamma$ which was required for (14). The point of this choice of λ is to maximize the lower bound inside the probability on the right-hand side of (14). So, continuing from (14) with Chebyshev's inequality and the previous lemma,

$$\begin{aligned}
&\mathbf{P}\{Q_a(t)\ge 2u+[V^\varrho t]\}\\
&\quad\le \mathbf{P}\left\{\widetilde{J}^\varrho(t)-\widetilde{J}^{V^\varrho,\lambda}(t)\ge \frac{u^2}{4t}(1+\varrho)^2\right\}+C_1\exp\left\{-C_2\frac{u^3}{t^2}\right\}\\
&\quad\le C_3\frac{t^2}{u^4}\mathbf{Var}(J^\varrho(t)-J^{V^\varrho,\lambda}(t))+C_1\exp\left\{-C_2\frac{u^3}{t^2}\right\}\\
&\quad\le C_3\frac{t^2}{u^4}(2\mathbf{Var}(J^\varrho(t)+2\mathbf{Var}J^{V^\varrho,\lambda}(t))+C_1\exp\left\{-C_2\frac{u^3}{t^2}\right\}\\
&\quad\le C_3\frac{t^2}{u^4}\cdot\mathbf{E}\big(|Q_a(t)-[V^\varrho t]|\big)+C_4\frac{t^2}{u^3}+C_5\frac{t^2}{u^4}+C_1\exp\left\{-C_2\frac{u^3}{t^2}\right\}.
\end{aligned}$$

Extension from integral u to real u is achieved by adjusting constants on the last line above.

Case 2 $bt\le u<2t$. Then $bu/2<bt$ and

$$\mathbf{P}\{Q_a(t)\ge 2u+[V^\varrho t]\}\le \mathbf{P}\{Q_a(t)\ge 2\cdot bu/2+[V^\varrho t]\}.$$

Case 1 can be applied with u replaced by $bu/2$, at the price of adjusting some constants with powers of b.

Case 3 $u\ge 2t$. Since $Q_a(t)$ is bounded above by a rate one Poisson process $N(t)$,

$$\begin{aligned}
\mathbf{P}\{Q_a(t)\ge 2u+[V^\varrho t]\}&\le \mathbf{P}\{Q_a(t)\ge 2u\}\le \mathbf{P}\{N(t)\ge 2u\}\\
&\le e^{-2u}\cdot\mathbf{E}(e^{N(t)})=e^{t(e-1)-2u}\le e^{-u}
\end{aligned}$$

for all t. Combining the bounds from the cases proves the lemma. □

We are ready to complete the proof of the upper bound of Theorem 2. Introduce the notation

$$\Psi(t)=\mathbf{E}\big(|Q_a(t)-[V^\varrho t]|\big).$$

In order to get a bound on $\mathbf{P}\{|Q_a(t)-[V^\varrho t]|\ge 2u\}$ we need a lower tail bound for $Q_a(t)$, similar to (18). This can be obtained by methods analogous to the ones we have applied throughout Sect. 3. In this case we have to couple a process with initial distribution $\underline{\hat{\mu}}_\sigma$ for some $\sigma>\varrho$ close enough to ϱ and derive similar lemmas to get the same conclusion for $\mathbf{P}\{Q_a(t)\le -2u+[V^\varrho t]\}$ as in (18) with readjusted constants.

Introduce a large constant $2<r<\infty$. For $t\ge 1$ and $u\ge rt^{2/3}$ we can combine (18) and the matching lower tail bound. Replace $2u$ by u (we made sure $u/2\ge 1$), then some algebra leads to

$$\begin{aligned}
&\mathbf{P}\{|Q_a(t)-[V^\varrho t]|\ge u\}\\
&\quad\le C_1\frac{t^2}{u^4}\Psi(t)+C_2(r)\left(\frac{t^2}{u^3}+\exp\left\{-C_3(r)\frac{u}{t^{2/3}}\right\}\right).
\end{aligned}\tag{19}$$

The two exponential terms in (18) were combined via $e^{-u}\le\exp(-ut^{-2/3})$ and $\exp(-C_2u^3t^{-2})\le\exp(-C_2r^2ut^{-2/3})$.

Let $1 \le m < 3$. Integrate bound (19) over $u \in [rt^{2/3}, \infty)$:

$$\begin{aligned}
&\mathbf{E}\big(|Q_a(t) - [V^\varrho t]|^m\big) \\
&\quad \le r^m t^{2m/3} + m \int_{rt^{2/3}}^{\infty} \mathbf{P}\{|Q_a(t) - [V^\varrho t]| \ge u\} u^{m-1}\, \mathrm{d}u \\
&\quad \le C_1 r^{m-4} \Psi(t) t^{2+(2/3)(m-4)} + C_4(r) t^{2m/3}.
\end{aligned} \tag{20}$$

The constant C_1 depends on m but not on r, while C_4 depends on both m and r. To get the final bounds, take first $m = 1$ in (20) to get

$$\Psi(t) \le C_1 r^{-3} \Psi(t) + C_4(r) t^{2/3}.$$

Since C_1 is independent of r, fixing r large enough gives $\Psi(t) \le C_5(r) t^{2/3}$. Using this bound in the last line of (20) implies

$$\mathbf{E}\big(|Q_a(t) - [V^\varrho t]|^m\big) \le C_6(r) t^{2m/3}$$

for $1 < m < 3$, which completes the proof of the upper bound of Theorem 2.

4 Lower Bound

The lower bound is proved by perturbing the stationary distribution of a process on a segment of the lattice. We start with discussing the initial distribution of some coupled processes.

4.1 Perturbing a Segment Initially

Recall again the characteristic speeds $V^\varrho = \frac{1}{(1+\varrho)^2}$ and $V^\lambda = \frac{1}{(1+\lambda)^2}$. We set $\varrho > \lambda$, hence $V^\varrho < V^\lambda$. Throughout this section $u > 0$ denotes a fixed positive integer, and

$$n = [V^\lambda t] - [V^\varrho t] + u.$$

Recall the Definition 7 of the measures ν and $\hat{\nu}$. Define an initial product distribution of two configurations $(\underline{\eta}(0), \underline{\zeta}(0))$ by setting the marginals on each lattice site i:

$$\begin{cases}
(\eta_i(0), \zeta_i(0)) \sim \nu & \text{if } i < -n, \\
(\eta_i(0),\ \zeta_i(0)) \sim \hat{\nu} & \text{if } i = -n, \\
\eta_i(0) = \zeta_i(0) \sim \mu_\lambda & \text{if } -n < i \le 0, \\
(\eta_i(0), \zeta_i(0)) \sim \nu & \text{if } i > 0.
\end{cases}$$

Note that the number of particles at site $-n$ is $\hat{\mu}_\lambda$-distributed for $\underline{\eta}(0)$. Except for this perturbation $\underline{\eta}(0)$ starts in the stationary Geometric product distribution $\underline{\mu}_\lambda$. The process $\underline{\zeta}(\cdot)$ initially has distribution $\underline{\mu}_\varrho$, except at sites $\{-n+1, \dots, 0\}$ where the parameter ϱ has been replaced by λ, and at site $-n$ where it has measure $\hat{\mu}_\varrho$.

We add a second class particle to the process $\underline{\eta}(.)$ initially started at site $-n$ and denote its position at time t by $Q^{(-n)}(t)$. Introduce, as before, $\underline{\eta}^+(t) := \underline{\eta}(t) + \underline{\delta}_{Q^{-n}(t)}$.

Define a third initial configuration by

$$\xi_i(0) = \begin{cases} \zeta_i(0) & \text{if } i \le -n, \\ \eta_i(0) & \text{if } i > -n. \end{cases}$$

Apply the basic coupling to obtain the joint evolution of all these processes. This guarantees the majorizations

$$\underline{\eta}(t) \le \underline{\xi}(t) \le \underline{\zeta}(t) \quad \text{and} \quad \underline{h}^{\zeta}(t) \le \underline{h}^{\xi}(t)$$

where the last inequality is for column heights.

As before in (3), denote the net particle currents by $J^{V,\eta}$ and $J^{V,\zeta}$ in the respective processes $\underline{\eta}(\cdot)$ and $\underline{\zeta}(\cdot)$. The first observation is that $Q^{(-n)}$ gives one-sided control over the difference of these currents.

Lemma 12 *For any* $V \in \mathbb{R}$

$$Q^{(-n)}(t) \le [Vt] \quad \textit{implies } J^{V,\zeta}(t) - J^{V,\eta}(t) \le 0.$$

Proof Denote the positions of the $\xi - \eta$ second class particles at time t by

$$\cdots \le Y_k(t) \le \cdots \le Y_{-2}(t) \le Y_{-1}(t) \le Y_0(t).$$

This order of the labels is again preserved over time. Initially $Y_0(0) \le -n = Q^{(-n)}(0)$, and the respective jump rates $\mathbb{I}_{\eta_{Y_0(t)}=0}$ and $\mathbb{I}_{\eta_{Q^{(-n)}(t)}=0}$ of the second class particles ensure $Y_0(t) \le Q^{(-n)}(t)$ for all later times. Indeed, from the time Y_0 has jumped on $Q^{(-n)}$ they always jump together. So, Y_0 cannot leave $Q^{(-n)}$ behind itself. Saying it another way, once $Q^{(-n)}(t) = Y_0(t)$, this property will hold forever.

Now note that there are no $\xi - \eta$ second class particles strictly to the right of $Q^{(-n)}(t)$ at time t. So, the height difference is zero to the right of $Q^{(-n)}(t)$. Also recall the majorization $\underline{h}^{\zeta}(t) \le \underline{h}^{\xi}(t)$. Thus under $\{Q^{(-n)}(t) \le [Vt]\}$ we have

$$0 = h^{\xi}_{[Vt]}(t) - h^{\eta}_{[Vt]}(t) \ge h^{\zeta}_{[Vt]}(t) - h^{\eta}_{[Vt]}(t) = J^{V,\zeta}(t) - J^{V,\eta}(t).$$

□

Let $\underline{\hat{\omega}}(\cdot)$ be a TAZRP started from the distribution $\underline{\hat{\mu}}_{\varrho}$ shifted n sites to the left, i.e. at site $-n$ the number of particles is $\hat{\mu}_{\varrho}$-distributed. The next lemma compares the distributions of $\underline{\zeta}$ and $\underline{\hat{\omega}}$.

Lemma 13 *Denote by* $\mathbf{P}^{\hat{\omega}}$ *and* $\mathbf{P}^{\zeta}$ *the probability of events that depend only on the respective processes* $\underline{\hat{\omega}}(\cdot)$ *and* $\underline{\zeta}(\cdot)$. *Then there exist* $\gamma > 0$ *such that for all* $\lambda > \varrho - \gamma$ *the following inequality holds*:

$$\mathbf{P}^{\zeta}(\cdot) \le \mathbf{P}^{\hat{\omega}}(\cdot)^{\frac{1}{2}} \cdot \exp\left[\frac{n(\varrho - \lambda)^2}{\varrho(1+\varrho)}\right].$$

Proof Let

$$Z = \sum_{i=-n+1}^{0} \zeta_i(0).$$

Z has a Negative Binomial$(n, \frac{1}{1+\lambda})$ distribution, namely

$$m^{\lambda}(z) := \mathbf{P}(Z = z) = \binom{n+z-1}{n-1}\left(\frac{\lambda}{1+\lambda}\right)^{z}\left(\frac{1}{1-\lambda}\right)^{n}$$

for $z \geq 0$. We use the Cauchy-Schwarz inequality below to perform a change of measure on this negative binomial distribution.

$$\begin{aligned}\mathbf{P}^{\zeta}(\cdot) &= \sum_{z=0}^{\infty}\mathbf{P}^{\zeta}(\cdot \mid Z = z)[m^{\varrho}(z)]^{\frac{1}{2}} \cdot \frac{m^{\lambda}(z)}{[m^{\varrho}(z)]^{\frac{1}{2}}} \\ &\leq \left[\sum_{z=0}^{\infty}[\mathbf{P}^{\zeta}(\cdot \mid Z = z)]^{2}\, m^{\varrho}(z)\right]^{\frac{1}{2}} \cdot \left[\sum_{z=0}^{\infty}\frac{[m^{\lambda}(z)]^{2}}{m^{\varrho}(z)}\right]^{\frac{1}{2}}.\end{aligned}$$

Now the last part equals $\sum_{z=0}^{\infty}\binom{n+z-1}{n-1}(\frac{\lambda^2(1+\varrho)}{(1+\lambda)^2\varrho})^{z} \cdot (\frac{(1+\varrho)}{(1+\lambda)^2})^{n}$. In this formula we can recognize the negative binomial mass function with parameter $0 < 1 - \frac{\lambda^2(\varrho+1)}{(1+\lambda)^2\varrho} < 1$, so we use the identity

$$\sum_{z=0}^{\infty}\binom{n+z-1}{n-1}\left(\frac{\lambda^2(1+\varrho)}{(1+\lambda)^2\varrho}\right)^{z} \cdot \left(1 - \frac{\lambda^2(1+\varrho)}{(1+\lambda)^2\varrho}\right)^{n} = 1$$

and the remaining factors equal $(1 - \frac{\lambda^2(\varrho+1)}{(1+\lambda)^2\varrho})^{-n} \cdot (\frac{\varrho+1}{(\lambda+1)^2})^{n} = [1 - \frac{(\varrho-\lambda)^2}{\varrho(1+\varrho)}]^{-n}$. So we get

$$\begin{aligned}\mathbf{P}^{\zeta}(\cdot) &\leq \left[\sum_{z=0}^{\infty}[\mathbf{P}^{\zeta}(\cdot \mid Z = z)]^{2}\, m^{\varrho}(z)\right]^{\frac{1}{2}} \cdot \left[1 - \frac{(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right]^{\frac{-n}{2}} \\ &\leq \left[\sum_{z=0}^{n}\mathbf{P}^{\zeta}(\cdot \mid Z = z)\, m^{\varrho}(z)\right]^{\frac{1}{2}} \cdot \exp\left[\frac{n(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right].\end{aligned}$$

In the last line we used the fact that $1 - x \geq e^{-2x}$ for all $0 < x \leq 0.5$. This will be satisfied for $x = \frac{(\varrho-\lambda)^2}{\varrho(\varrho+1)}$, for λ close enough to ϱ. All that is left is to recognize that $\mathbf{P}^{\zeta}(\cdot \mid Z = z)$ is the probability depending on a process $\underline{\zeta}(\cdot)$ whose initial distribution coincides with the distribution of $\hat{\omega}$ outside $\{-n+1\ldots 0\}$, with z particles distributed in that interval with each configuration equally likely. Namely, each configuration with z particles in this interval has probability $(\frac{\lambda}{1+\lambda})^{z}(\frac{1}{1+\lambda})^{n}$, so conditioned on the event that z particles are here initially, each configuration has probability $\frac{1}{\binom{n+z-1}{n-1}}$. Summing these conditionals, weighted with the Negative Binomial$(n, \frac{1}{1+\varrho})$ mass function $m^{\varrho}(z)$, gives the product Geometric initial distribution $\underline{\mu}_{\varrho}$ of the process $\underline{\hat{\omega}}(\cdot)$. $\square$

4.2 Proof of the Lower Bounds

Now we only need two steps to obtain the lower bound. In this part we need again the concept of a second class antiparticle started from the origin on a $\underline{\mu}_{\varrho}$-stationary process $\underline{\omega}(\cdot)$ initially perturbed by setting $\omega_0(0) \sim \hat{\mu}_{\varrho} + \delta_0$. We denote its position at time t by $Q_a(t)$. The quantity of primary interest is abbreviated, as before, by $\Psi(t) = \mathbf{E}(|Q_a(t) - [V^{\varrho}t]|)$.

The first step is proving an upper bound on $Q^{(-n)}(t)$. As before u is an arbitrary but fixed positive integer and $n = [V^\lambda t] - [V^\varrho t] + u$.

Lemma 14

$$\mathbf{P}\{Q^{(-n)}(t) > [V^\varrho t]\} \le \frac{\Psi(t)}{u} + \frac{4t(\varrho-\lambda)}{u} + \frac{2}{u}. \tag{21}$$

Proof Below $Q^\lambda(t)$ stands for the position of a second class particle started from the origin on a process $\underline{\hat{\eta}}(\cdot)$ in $\underline{\hat{\mu}}_\lambda$ distribution. Translation invariance implies

$$\begin{aligned}
&\mathbf{P}\{Q^{(-n)}(t) > [V^\varrho t]\}\\
&\quad = \mathbf{P}\{Q^{(-n)}(t) + n - [V^\lambda t] > u\}\\
&\quad = \mathbf{P}\{Q^\lambda(t) - [V^\lambda t] > u\}\\
&\quad \le \frac{\mathbf{E}(|Q^\lambda(t) - [V^\lambda t]|)}{u}\\
&\quad \le \frac{\mathbf{E}(|Q^\lambda(t) - Q_a(t)|)}{u} + \frac{\mathbf{E}(|Q_a(t) - [V^\varrho t]|)}{u} + \frac{[V^\lambda t] - [V^\varrho t]}{u}.
\end{aligned}$$

As in Lemma 10, the first term equals $\frac{t}{u}(\frac{1}{(1+\lambda)^2} - \frac{1}{(1+\varrho)^2})$, bounded from above by $\frac{2}{u}t(\varrho-\lambda)$ after some calculations. The second term is $\Psi(t)/u$, and the third term is similarly estimated by $\frac{2}{u}t(\varrho-\lambda) + \frac{2}{u}$, the last part coming from possible integer part errors. □

The second step is an estimate of the probability of the complement of the event in (21).

Lemma 15 *For any* $0 < K < t\frac{(\varrho-\lambda)^2}{(1+\varrho)^3} - 2\varrho$,

$$\begin{aligned}
\mathbf{P}\{Q^{(-n)}(t) \le [V^\varrho t]\} \le{}& \frac{(2\varrho(1+\varrho))^{1/2}(6+\Psi(t))^{1/2}}{t\frac{(\varrho-\lambda)^2}{(1+\varrho)^3} - 2\varrho - K} \cdot \exp\left[\frac{n(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right]\\
&+ \frac{2\varrho(1+\varrho)(6+\Psi(t))}{K^2} + \frac{4t(\varrho-\lambda)}{K^2}.
\end{aligned}$$

Proof Lemma 12 leads to

$$\begin{aligned}
\mathbf{P}\{Q^{(-n)}(t) \le [V^\varrho t]\} &\le \mathbf{P}\{J^{V^\varrho,\zeta}(t) - J^{V^\varrho,\eta}(t) \le 0\}\\
&\le \mathbf{P}\left\{J^{V^\varrho,\zeta}(t) \le K + t\left(\frac{\lambda}{1+\lambda} - \frac{\lambda}{(1+\varrho)^2}\right) + 2\lambda\right\} \qquad (22)\\
&\quad + \mathbf{P}\left\{J^{V^\varrho,\eta}(t) > K + t\left(\frac{\lambda}{1+\lambda} - \frac{\lambda}{(1+\varrho)^2}\right) + 2\lambda\right\} \qquad (23)
\end{aligned}$$

We apply Lemma 13 to line (22) to bound it by the probability of the process $\hat{\omega}$:

$$
\begin{aligned}
&\leq \left[\mathbf{P}^{\hat{\omega}}\left\{J^{\varrho}(t) \leq K + t\left(\frac{\lambda}{1+\lambda} - \frac{\lambda}{(1+\varrho)^2}\right) + 2\lambda\right\}\right]^{\frac{1}{2}} \cdot \exp\left[\frac{n(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right] \\
&\leq \left[\mathbf{P}^{\hat{\omega}}\left\{\widetilde{J}^{\varrho}(t) \leq K - t\frac{(\varrho-\lambda)^2}{(1+\lambda)(1+\varrho)^2} + 2\lambda\right\}\right]^{\frac{1}{2}} \cdot \exp\left[\frac{n(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right] \\
&\leq \left[\mathbf{P}^{\hat{\omega}}\left\{\widetilde{J}^{\varrho}(t) \leq K - t\frac{(\varrho-\lambda)^2}{(1+\varrho)^3} + 2\varrho\right\}\right]^{\frac{1}{2}} \cdot \exp\left[\frac{n(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right] \\
&\leq \frac{[\mathbf{Var}(J^{\varrho}(t))]^{1/2}}{t\frac{(\varrho-\lambda)^2}{(1+\varrho)^3} - 2\varrho - K} \cdot \exp\left[\frac{n(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right] \\
&\leq \left\{\frac{(2\varrho(1+\varrho))^{1/2}(6+\Psi(t))^{1/2}}{t\frac{(\varrho-\lambda)^2}{(1+\varrho)^3} - 2\varrho - K} + \frac{\sqrt{2}}{t\frac{(\varrho-\lambda)^2}{(1+\varrho)^3} - 2\varrho - K}\right\} \cdot \exp\left[\frac{n(\varrho-\lambda)^2}{\varrho(1+\varrho)}\right]. \qquad (24)
\end{aligned}
$$

When centering, we used $J^{\varrho}(t) = \widetilde{J}^{\varrho}(t) + [\mathbf{E}J^{\varrho}(t) - \mathbf{E}^{\varrho}J^{\varrho}(t)] + [\mathbf{E}^{\varrho}J^{\varrho}(t) - t(\frac{\varrho}{1+\varrho} - \frac{\varrho}{(1+\varrho)^2})] + t(\frac{\varrho}{1+\varrho} - \frac{\varrho}{(1+\varrho)^2})$, where both error terms in the brackets $[\cdot]$ are nonnegative. So, $J^{\varrho}(t) \leq K + t(\frac{\lambda}{1+\lambda} - \frac{\lambda}{(1+\varrho)^2}) + 2\lambda$ implies $\widetilde{J}^{\varrho}(t) \leq K - t\frac{(\varrho-\lambda)^2}{(1+\lambda)(1+\varrho)^2} + 2\lambda$. Then, in the last line we used Lemma 10 to bound the variance by the function $\Psi(t)$ even though the second class particle in $\hat{\omega}$ starts at $-n$ rather than at the origin, namely $\mathbf{Var}(J^{\varrho}(t)) \leq 2 + 12\varrho(1+\varrho) + 2\varrho(1+\varrho)\Psi(t)$. We use the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ in order to get rid of the term $+2$. The only change needed in the proof of Lemma 10 is in the calculation (17) where we must add $\mathbf{E}(\omega_{-n}(0)^2)$ instead of $\mathbf{E}(\omega_0(0)^2)$, but these are equal.

A similar coupling consideration shows that $\mathbf{E}(J^{V^{\varrho},\eta}(t))$ differs by at most λ from the same expectation taken under a stationary $\underline{\mu}_{\lambda}$ initial condition. Thus taking integer parts again into account, giving another error term λ, line (23) is bounded from above by

$$
\mathbf{P}\{\widetilde{J}^{V^{\varrho},\eta}(t) > K\} \leq \frac{\mathbf{Var}(J^{V^{\varrho},\eta})}{K^2}.
$$

Lemma 10 can be applied again to bound this variance with the similar change of shifting all processes by n sites to the left. Hence we can continue from above to bound line (23) with

$$
\frac{2\varrho(1+\varrho)(6+\Psi(t))}{K^2} + \frac{4t(\varrho-\lambda)}{K^2}.
$$

□

Now we are at the last step of proving the lower bound of Theorem 2. By Jensen's inequality it suffices to prove for the case $m = 1$, in other words that

$$
\liminf_{t\to\infty} t^{-2/3}\Psi(t) > 0.
$$

In the last two lemmas take

$$
u = \lceil ht^{2/3} \rceil, \qquad \varrho - \lambda = bt^{-1/3}, \quad \text{and} \quad K = bt^{1/3},
$$

where h and b are large, in particular b large enough to have $bt^{\frac{1}{3}} < \frac{b^2}{(1+\varrho)^3}t^{\frac{1}{3}} - 2\varrho$ so that K satisfies the assumption of Lemma 15. Then

$$n = [V^{\lambda}t] - [V^{\varrho}t] + u \le \frac{2+\varrho+\lambda}{(1+\lambda)^2(1+\varrho)^2}(\varrho-\lambda)t + 2 + u \le (2b+h)t^{2/3} \le Ct^{2/3}$$

for large enough t. We can simplify the outcomes of Lemmas 14 and 15 to the inequalities

$$\mathbf{P}\{Q^{(-n)}(t) > [V^{\varrho}t]\} \le C\frac{\Psi(t)}{t^{2/3}} + \frac{4b}{h} + \frac{2}{ht^{2/3}} \tag{25}$$

and

$$\mathbf{P}\{Q^{(-n)}(t) \le [V^{\varrho}t]\} \le C\left(\frac{6+\Psi(t)}{t^{2/3}}\right)^{1/2} + C\frac{6+\Psi(t)}{t^{2/3}} + \frac{4}{b} + \frac{C}{t^{1/3}}. \tag{26}$$

The new constant C depends on b and h.

The lower bound now follows because the left-hand sides of (25)–(26) add up to one for each fixed t, while we can fix b large enough and then h large enough so that $4b/h + 4/b < 1$. Then $t^{-2/3}\Psi(t)$ must have a positive lower bound for all large enough t. This completes the proof of Theorem 2.

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